

BASE TREE PROPERTY

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ABSTRACT. Building on previous work of Balcar, Pelant and Simon we investigate σ -closed partial orders of size continuum. We provide both an internal and external characterization of such partial orders by showing that (1) every σ -closed partial order of size continuum has a base tree and that (2) σ -closed forcing notions of density \mathfrak{c} correspond exactly to regular suborders of the collapsing algebra $\text{Coll}(\omega_1, 2^\omega)$.

We further study some naturally occurring examples of such partial orders.

INTRODUCTION

A partially ordered set (P, \leq) is σ -closed if every countable decreasing sequence of elements of P has a lower bound. In this note we study σ -closed partial orders of size continuum. Orders of this type naturally arise in combinatorial and descriptive set-theory, topology and analysis.

An essential example is the *collapsing algebra* $\text{Coll}(\omega_1, 2^\omega)$, i.e. the completion, in the sense of Boolean algebra, of the complete binary tree of height ω_1 . This forcing notion has several presentations:

- $(\text{Fn}(\omega_1, \{0, 1\}, \omega_1), \supseteq)$ - ordering for adding a new subset of ω_1 ,
- $(\text{Fn}(\omega_1, \mathbb{R}, \omega_1), \supseteq)$ - ordering for the consistency of the continuum hypothesis,
- $(\text{Fn}(2^\omega, \{0, 1\}, \omega_1), \supseteq)$ - ordering for adding \mathfrak{c} -many subsets of ω_1 ,
- the natural ordering for adding a \diamond -sequence,
- Jech's forcing for adding a Suslin tree by countable conditions.

All these orderings are forcing equivalent, in fact, they have isomorphic base trees (see Theorem 2.1 for the term base tree).

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Consider now the set $[\omega]^\omega$ of all infinite sets of natural numbers ordered by inclusion. This order is not σ -closed, but it is also not *separative*.¹ The separative quotient of $([\omega]^\omega, \subseteq)$ are the positive elements in the Boolean algebra $\mathcal{P}(\omega)/fin$. In [1] the surprising fact that also $\mathcal{P}(\omega)/fin$ has a base tree was established. It was then studied in [8], [10], [12].

Since then many other naturally occurring examples were studied ([2],[3]) and in each case the methods of [1] were used to prove the corresponding Base Tree Theorem.

In this note we prove this general fact for all partial orders with a dense σ -closed subset of size continuum. We also identify the σ -closed forcings of size continuum as the regular subalgebras of the collapsing algebra $Coll(\omega_1, 2^\omega)$.

We then present some of the standard examples and review the relevant published results.

1. MAIN RESULTS

The *height* of a partial order (P, \leq) , $\mathfrak{h}(P)$ shortly, is the minimal cardinality of a system of open dense subsets of P such that the intersection of the system is not dense; i.e. $\mathfrak{h}(P) = \min\{|H| : \forall D \in H (D \text{ is open dense}) \wedge \bigcap H \text{ is not dense}\}$. For a Boolean algebra B we define $\mathfrak{h}(B)$ as the height of the ordering $(B \setminus \{0\}, \leq)$, where \leq is the canonical ordering on B . If B is complete, it coincides with its *distributivity number*. We will deal mostly with non-atomic orderings but for completeness we allow atomic orderings in the definition too. Thus, if (P, \leq) is *atomic*, i.e. there is a set of minimal elements such that every other element is above one of them, then we set $\mathfrak{h}(P) = \infty$.

The height is a forcing invariant, that means every dense subset of an ordering has the same height. In particular, $\mathfrak{h}(P) = \mathfrak{h}(\text{RO}(P))$.

Fact 1.1. *For an ordering P , $\mathfrak{h}(P)$ is the minimal cardinal κ such that forcing with P adds a new subset of κ . In particular, forcing with P preserves all cardinals less than κ .*

An ordering P is *homogeneous in \mathfrak{h}* (*homogeneous in height*) if for every $p \in P$ $\mathfrak{h}(\downarrow p) = \mathfrak{h}(P)$. The following proposition shows that every partial order can be decomposed into factors homogeneous in density. For complete Boolean algebras there is a canonical such decomposition.

¹Recall that a partial order P is separative if whenever p, q are elements of P such that $p \not\leq q$, there is an $r \in P$ such that $r \leq p$ and $r \perp q$.

Proposition 1.2. *Let B be a complete Boolean algebra. Then $B \cong \prod_{b \in I} B \restriction b$, where I is a partition of unity and $B \restriction b$ is homogeneous in the height for every $b \in I$.*

Moreover, $\mathfrak{h}(B \restriction a) \neq \mathfrak{h}(B \restriction b)$ if $a \neq b$ for $a, b \in I$.

Proof. Let A be the set of all atoms, $\uparrow A = \{b \in B : \exists a \in A (a \leq b)\}$, then $B \restriction \bigvee \uparrow A$ is the first factor homogeneous in the height ∞ .

Next, we work with an atomless complete algebra $B_0 = B \restriction (-\bigvee \uparrow A)$ ($B \cong B_0 \times B \restriction \bigvee \uparrow A$). Let $(D_\alpha)_{\alpha < \mathfrak{h}(B_0)}$ be the system of open dense subsets of B_0 such that $\bigcap_{\alpha < \mathfrak{h}(B_0)} D_\alpha$ is not dense. Let A_1 be the subset of elements of B_0 witnessing the non-density, i.e. $\downarrow a \cap \bigcap_{\alpha < \mathfrak{h}(B_0)} D_\alpha = \emptyset$ for every $a \in A_1$. We claim that for every $a \in A_1$ $B \restriction a$ is homogeneous in the height (with height $\mathfrak{h}(B_0)$). Assume not, then there is some $a \in A_1$ and $b < a$ such that $\mathfrak{h}(B \restriction b) < \mathfrak{h}(B \restriction a)$. Thus, there is a system $(S_\alpha)_{\alpha < \mathfrak{h}(B \restriction b)}$ of open dense subsets of $B \restriction b$ with a non-dense intersection below b . But if we set $D_\alpha = S_\alpha \cup B_0 \setminus \downarrow b$ then we get a system of open dense subsets in B_0 without a dense intersection less than $\mathfrak{h}(B_0)$, that is a contradiction.

We take a join $\bigvee A_1$ of all elements from A_1 and the factor $B \restriction \bigvee A_1$ is homogeneous in the height. We continue with the remainder $B_1 = B_0 \restriction (-\bigvee A_1)$ and by the same way get a set A_2 of elements witnessing the non-density of the intersection of a system of open dense subsets of size $\mathfrak{h}(B_1)$. It is possible that $\mathfrak{h}(B_1) = \mathfrak{h}(B_0)$. In this case, we join elements of A_2 with elements of A_1 . In the opposite case, $B_1 \restriction \bigvee A_2$ is a new factor homogeneous in the height.

We continue similarly until we treat all elements of B . We end up with the desired decomposition. \square

Definition 2 (Base tree property). An ordering (P, \leq) has the base tree property (we shall shortly say it has the *BT-property*) if it contains a dense subset $D \subseteq P$ with the following three properties:

- it is atomless; i.e. for every $d \in D$ there are elements $d_1, d_2 \in D$ below d such that $d_1 \perp d_2$
- it is σ -closed
- $|D| \leq \mathfrak{c}$

It can be easily seen that assuming the Continuum Hypothesis, all partial orders with the BT-property are forcing equivalent with $\text{Coll}(\omega_1, 2^\omega)$ and, consequently have a tree base. In fact, the following is true in ZFC.

Theorem 2.1 (The base tree theorem). *Let (P, \leq) be an ordering homogeneous in the height with the BT-property. Then there are $\mathfrak{h}(P)$ maximal antichains $(T_\alpha)_{\alpha < \mathfrak{h}(P)} \subseteq P$ such that:*

- (i) $(T = \bigcup_{\alpha < \mathfrak{h}} T_\alpha, \geq)$ is a tree of height $\mathfrak{h}(P)$, where T_α is the α -th level of the tree,
 - (ii) each $t \in T$ has \mathfrak{c} immediate successors,
 - (iii) T is dense in P .
- T is called the base tree of P .

Proof. We need to work with some dense subset guaranteed by the definition of the BT-property rather than with P itself. To avoid introducing next new symbols and sets, we assume P itself has the properties.

We use the definition of the height. So we have a system $(A_\alpha)_{\alpha < \mathfrak{h}(P)}$ of open dense subsets with a non-dense intersection. We need to ensure the intersection to be empty. Suppose $\bigcap_{\alpha < \mathfrak{h}(P)} A_\alpha$ is not empty. Since for each $a \in \bigcap_{\alpha < \mathfrak{h}(P)} A_\alpha$ $\mathfrak{h}(\downarrow a) = \mathfrak{h}(P)$ we have a system $(\bar{A}_\alpha)_{\alpha < \mathfrak{h}(P)}$ of open dense sets below a of the same size such that their intersection is non-dense below a . For each such a we replace $\downarrow a \cap A_\alpha$ by \bar{A}_α (i.e. $(A_\alpha \setminus \downarrow a) \cup \bar{A}_\alpha$). We get a new system of open dense subsets with a non-dense intersection. If this intersection is again non-empty we repeat the same procedure for each element from the intersection. We repeat this procedure as long as necessary to get the system $(B_\alpha)_{\alpha < \mathfrak{h}(P)}$ of open dense subsets with an empty intersection.

Next, we extract from each open dense set B_α a maximal antichain C_α . We claim that for every $p \in P$ there is at least one maximal antichain C_α and elements $a, b \in C_\alpha$ such that p is compatible with both of them. Suppose that for some $p \in P$ and for every $\alpha < \mathfrak{h}(P)$ there is only one element c_α from C_α that is compatible with p . But then p is in fact below c_α (since if $p \not\leq c_\alpha$ then there is a $p_0 \leq p$ that is disjoint with c_α but necessarily compatible with another element of C_α). But this means that $p \in \bigcap_{\alpha < \mathfrak{h}(P)} \downarrow C_\alpha \subseteq \bigcap_{\alpha < \mathfrak{h}(P)} B_\alpha$ and that is a contradiction with the fact the intersection is empty.

Before constructing the levels of T we modify the antichains into more suitable form, more accurately we modify them into the system $(D_\alpha)_{\alpha < \mathfrak{h}(P)}$ where D_β refines D_α if $\alpha < \beta$. This can be easily done if we set D_α to be the common refinement of $(C_\gamma)_{\gamma \leq \alpha}$ and $(D_\gamma)_{\gamma < \alpha}$.

The levels of the tree T will be the maximal antichains. What we need to care is to ensure that T is dense and that every element of T has \mathfrak{c} immediate successors. We begin by showing that for each element $p \in P$ there is an antichain D_α with \mathfrak{c} -many elements compatible with p . There is some D_{α_0} and elements $d_0, d_1 \in D_{\alpha_0}$ compatible with p , i.e. there are elements $p_0 \leq d_0, p_1 \leq d_1$ below p . Then again there is some D_{α_1} and elements $d_{00}, d_{01}, d_{10}, d_{11} \in D_{\alpha_1}$, the first two compatible with p_0 , the last two with p_1 (note that this is the place where we need

the antichains to be refining; since in general there would be some D_{β_1} with compatible elements with p_0 and some D_{β_2} with compatible elements with p_1 but in our case we can take α_1 to be $\sup\{\beta_1, \beta_2\}$. We again get $p_\zeta \leq p$ for each $\zeta \in {}^2\{0, 1\}$. We continue until we get an appropriate $p_\zeta \leq p$ for each $\zeta \in {}^{<\omega}\{0, 1\}$. For every $\xi \in {}^\omega\{0, 1\}$ we have a descending chain $p \geq p_{\xi \upharpoonright \{0\}} \geq \dots p_{\xi \upharpoonright n} \geq \dots$ with a lower bound (due to σ -closedness) p_ξ . $p_{\xi_1} \perp p_{\xi_2}$ for $\xi_1 \neq \xi_2$. Thus we see that there is a maximal antichain of size \mathfrak{c} below p ; we denote it $\mathcal{A}(p)$. Each such p_ξ is compatible with some element d_ξ of D_α where $\alpha = \sup\{\alpha_n : n \in \omega\}$. And again $\xi_1 \neq \xi_2$ implies $d_{\xi_1} \neq d_{\xi_2}$.

Let $P_\alpha = \{p \in P : p \text{ is compatible with } \mathfrak{c}\text{-many elements of } D_\alpha\}$. We see that $P = \bigcup_{\alpha < \mathfrak{h}(P)} P_\alpha$. Since $|P_\alpha| \leq \mathfrak{c}$ for each α there is an injective mapping $f_\alpha : P_\alpha \rightarrow {}^\omega 2$ such that $p_{f_\alpha(p)} \leq p$ for every $p \in P_\alpha$, where $p_{f_\alpha(p)}$ is from the construction above.

Now we are ready to start the construction. We set $T_0 = D_0$ and for each $\alpha + 1$ we set $T_{\alpha+1}$ to be the common refinement of $D_{\alpha+1}$, $\mathcal{A}(p)$ for all $p \in T_\alpha$ and $\{p_{f_\alpha(p)} : p \in P_\alpha\}$. For α limit, T_α is just the common refinement of $(T_\gamma)_{\gamma < \alpha}$.

Note that by refining $\mathcal{A}(p)$ for all $p \in T_\alpha$ we ensure that each element of the tree has \mathfrak{c} -many immediate successors and by refining $\{p_{f_\alpha(p)} : p \in P_\alpha\}$ that T is dense. This finishes the proof. \square

Corollary 2.2. *The following statements for an ordering (P, \leq) are equivalent:*

- (i) P has the BT-property,
- (ii) P has a dense subset with the BT-property,
- (iii) Every dense subset of P has the BT-property,
- (iv) $\text{RO}(P)$ has the BT-property.

Proof. Note that (i) \Rightarrow (ii) and (iii) \Rightarrow (iv) follow from the definition. It suffices to prove (ii) \Rightarrow (iii), (iv) \Rightarrow (i) is then a consequence.

We need to find a dense subset of a given dense subset that is moreover σ -closed. Atomlessness is determined by the whole ordering, the restriction on size will be clear.

Assuming (ii), we have a base tree T , we are given a dense subset D and we show there is a σ -closed dense subset $S \subseteq D$.

We make S from maximal antichains. For every $t \in T_0$ we find a maximal antichain $A_t \subseteq D$ below t . $\bigcup_{t \in T_0} A_t$ is a first maximal antichain S_0 .

Then for every $s \in S_0$ we find a maximal antichain $M_s \subseteq T$ below s . Let $P_1 \subseteq T$ be a maximal antichain from T refining $\bigcup_{s \in S_0} A_s$ and T_1 . We again for every $p \in P_1$ find a maximal antichain $A_p \subseteq D$ from D , the union $\bigcup_{p \in P_1} A_p$ is S_1 .

Next isolated steps are treated similarly. We need not to omit P_α to be refining the tree level T_α . Then we refine it to $S_\alpha \subseteq D$.

For a limit α we take a refinement P_α of all P_β 's for $\beta < \alpha$ (which is also a refinement of S_β 's) and of T_α . Then we again refine it to $S_\alpha \subseteq D$.

The resulting set $S = \bigcup_{\alpha < \mathfrak{h}(P)} S_\alpha$ is dense and σ -closed. We ensured denseness by refining all levels of T . For σ -closedness observe that for every countable descending chain $s_0 \geq s_1 \geq \dots$ from S , where $s_n \in S_{\alpha_n}$, there is an inserted descending chain $p_0 \geq p_1 \geq \dots$ such that $p_0 \geq s_0 \geq p_1 \geq s_1 \geq \dots$, where $p_n \in P_{\alpha_n}$. This inserted chain has a lower bound p in P_α , where $\alpha = \sup\{\alpha_n : n \in \omega\}$, and p has some successor $s \in S$. \square

\square

In other words, having a σ -closed dense set is preserved by forcing equivalence among separative partial orders of size continuum. On the other hand, Zapletal in [13] has constructed a model in which the Continuum Hypothesis holds and there are two forcing equivalent separative partial orders of size \aleph_2 one σ -closed and the other without a σ -closed dense set. One has to wonder whether such a pair exists in ZFC.

Question 2.3. *Are there, in ZFC, two forcing equivalent separative partial orders, such that one is σ -closed and the other does not have a σ -closed dense set?*

Finally, using this internal characterization of the partial orders with the BT-property one can easily deduce the following external characterization.

Theorem 2.4. *Let (P, \leq) be an arbitrary ordering with the BT-property. Then $\text{RO}(P)$ is a regular subalgebra of $\text{Coll}(\omega_1, \mathfrak{c})$.*

Proof. Let $D \subseteq P$ be its dense subset witnessing the BT-property. Then $D \times \text{Fn}(\omega_1, \{0, 1\}, \omega_1)$ with induced Cartesian ordering clearly has the BT-property. And the height is ω_1 , thus it determines the complete Boolean algebra $\text{Col}(\omega_1, \mathfrak{c})$. Note that there is a regular embedding $e : D \rightarrow D \times \text{Fn}(\omega_1, \{0, 1\}, \omega_1)$ defined as $e(d) = (d, 1)$ where 1 is the biggest element in $\text{Fn}(\omega_1, \{0, 1\}, \omega_1)$, i.e. the empty set. e is extended to $\bar{e} : \text{RO}(P) \rightarrow \text{Col}(\omega_1, \mathfrak{c})$ mapping $\text{RO}(P)$ on a regular subalgebra of $\text{Col}(\omega_1, \mathfrak{c})$. \square

3. CLASSICAL EXAMPLES

The Boolean algebra $\mathcal{P}(\omega)/\text{fin}$ is a prototype of an ordering with the BT-property. Recall the definitions of the cardinal invariants $\mathfrak{p}, \mathfrak{t}, \mathfrak{h}$ [6].

The second fundamental example is $(\text{Dense}(\mathbb{Q}), \subseteq)$, where $\text{Dense}(\mathbb{Q})$ is a set of all dense subsets in rationals. The situation here is similar with the previous example, it is not separative and the ordering $(\text{Dense}(\mathbb{Q}), \subseteq)$ itself does not satisfy the BT-property. We move to the separative modification. The separative modification is $(\text{Dense}(\mathbb{Q}), \subseteq_{\text{nwd}})$, where $A \subseteq_{\text{nwd}} B$ if $A \setminus B$ is nowhere dense in \mathbb{Q} , has the BT-property. This ordering is studied in [2].

Let $\mathfrak{p}_{\mathbb{Q}}, \mathfrak{t}_{\mathbb{Q}}, \mathfrak{h}_{\mathbb{Q}}$ be the cardinal invariants of $(\text{Dense}(\mathbb{Q}), \subseteq_{\text{nwd}})$ defined in the same way as their counterparts in $([\omega]^\omega, \subseteq^*)$. It was proved in [2] that $\mathfrak{p}_{\mathbb{Q}} = \mathfrak{p}$ and $\mathfrak{t}_{\mathbb{Q}} = \mathfrak{t}$ whereas $\mathfrak{h}_{\mathbb{Q}}$ and \mathfrak{h} are incomparable in ZFC, $\mathfrak{h}_{\mathbb{Q}} < \mathfrak{h}$ and $\mathfrak{h}_{\mathbb{Q}} > \mathfrak{h}$ are both consistent (see [2] and [7]); and $\mathfrak{h}_{\mathbb{Q}} = \mathfrak{h}$ too of course.

For the third example, let \mathbb{A} be the *Cantor algebra*, i.e. the algebra of all clopen subset of 2^ω , and consider the countable product \mathbb{A}^ω modulo the ideal $\text{Fin} \subseteq \mathbb{A}^\omega$, where $\text{Fin} = \{f \in \mathbb{A}^\omega : |\{n : f(n) \neq 0\}| < \omega\}$. It satisfies the BT-property, moreover, $\mathbb{A}^\omega/\text{Fin}$ is homogeneous.

$\mathfrak{t}(\mathbb{A}^\omega/\text{Fin}) = \mathfrak{t}$ and $\mathfrak{h}(\mathbb{A}^\omega/\text{Fin}) \leq \min\{\mathfrak{h}, \text{add}(\mathcal{M})\}$ ([3]) and it is consistent that $\mathfrak{h}(\mathbb{A}^\omega/\text{Fin}) < \mathfrak{h}$ ([3],[9]).

For any Boolean algebra B let us consider an infinite product B^ω . Let J be an ideal on ω . By $\mathcal{I}_J \subseteq B^\omega$ we denote the ideal $\{f \in B^\omega : \{n \in \omega : f(n) \neq 0\} \in J\}$. The quotient algebra B^ω/\mathcal{I}_J consists of equivalence classes where $f, g \in B^\omega$ are equivalent if $\{n : f(n) \neq g(n)\} \in J$ ($f \triangle g \in \mathcal{I}_J$ equivalently). We state and prove a simple criterion for when such a product has the BT-property.

Theorem 3.1. *Let B be a Boolean algebra and J an ideal on ω . Then the reduced product B^ω/\mathcal{I}_J has the BT-property if and only if B contains a dense subset of size \mathfrak{c} and either $\mathcal{P}(\omega)/J$ is σ -closed or J is a maximal ideal and B is atomless.*

Proof. Since B^ω/\mathcal{I}_J contains a dense subset of size \mathfrak{c} if and only if B contains a dense subset of size less or equal to \mathfrak{c} the requirement on the cardinality is satisfied.

Suppose that $\mathcal{P}(\omega)/J$ is not σ -closed. Let $(X_n)_{n \in \omega}$ be a descending chain of infinite subsets of ω such that the chain $([X_n])_{n \in \omega}$ does not have a lower bound in $\mathcal{P}(\omega)/J$, where $[X_n]$ is the equivalence class containing X_n . We define the descending chain $([f_n])_{n \in \omega} \subseteq B^\omega/\mathcal{I}_J$ as follows: $f_n(i) = 1$ if $i \in X_n$ and $f_n(i) = 0$ otherwise (it is the image of the chain $([X_n])_{n \in \omega}$ via the regular embedding of $\mathcal{P}(\omega)/J$ into B^ω/\mathcal{I}_J). Suppose that it has a lower bound $[f]$. Then the support of f , i.e. the set $\{i : f(i) \neq 0\}$, would determine a lower bound for $([X_n])_{n \in \omega}$.

Next we use the fact mentioned in [3] that B^ω/\mathcal{I}_J can be written as an iteration $\mathcal{P}(\omega)/J \star B^\omega/\dot{\mathcal{U}}$, where $\dot{\mathcal{U}}$ is a name for an ultrafilter added by $\mathcal{P}(\omega)/J$. For $[f] \in B^\omega/\mathcal{I}_J$ We define $\Phi([f]) = (\{i : f(i) \neq 0\}, [\dot{f}])$, where $[\dot{f}]$ is a name for an equivalence class containing f in $B^\omega/\dot{\mathcal{U}}$. Φ is easily verified to be a dense embedding which proves the fact.

Now observe that an ultrapower of any Boolean algebra is σ -closed. For a countable descending chain we can choose representants of equivalence classes $(f_n)_{n \in \omega}$ so that $\text{support } f_0 = \omega$, $\text{support } f_1 \supseteq \text{support } f_2 \supseteq \text{support } f_3 \supseteq \dots$ and $\bigcap_{n \in \omega} \text{support } f_n = \emptyset$ since the ultrafilter is non-principal. Then we set $f(i) = f_n(i)$ if n is the smallest number such that $i \in \text{support } f_n \setminus \text{support } f_{n+1}$. f clearly determines the lower bound for the chain. Hence, we conclude that B^ω/\mathcal{I}_J is σ -closed since an iteration of two σ -closed forcings is.

To check atomlessness, if J is not maximal then for any $f \in B^\omega$, where the support of f is not in J , we can always split the support of f into two disjoint infinite sets both outside of J , restrict f on these sets and make two disjoint elements of B^ω/\mathcal{I}_J below $[f]$. This is no longer possible in case J is a maximal ideal. For such a case we required B to be atomless and we find two disjoint successors coordinatewise. \square

4. ORDERINGS MADE OF IDEALS

We shall deal with orderings that consist of ideals on ω of some concrete type ordered by reverse inclusion.

For an illustration let us consider the following simple example.

4.1. Non-tall ideals. An ideal \mathcal{I} on ω is tall if for every $X \in [\omega]^\omega$ there is infinite $Y \subseteq X$ that belongs to \mathcal{I} . Consider the set \mathfrak{T} of all non-tall ideals on ω ordered by reverse inclusion.

At first, this ordering is not separative. However, for every $A \in [\omega]^\omega$ consider the ideal I_A of all subsets of ω that have a finite intersection with A . I_A is a non-tall ideal and $B \subseteq^* A$ implies $I_B \supseteq I_A$. Moreover, for every non-tall ideal \mathcal{I} and some infinite set A almost disjoint with every element of \mathcal{I} , $I_A \supseteq \mathcal{I}$. Thus we see that $([\omega]^\omega, \subseteq^*)$ is isomorphic with a dense subset of $(\mathfrak{T}, \supseteq)$ and of its separative modification showing that the separative modification of $(\mathfrak{T}, \supseteq)$ has the BT-property, however it is forcing equivalent to $([\omega]^\omega, \subseteq^*)$.

4.2. Summable ideals. The study of summable ideals is in fact mainly the study of sequences because we approach summable ideals via sequences in most cases. We shall focus on an ordering $(c_0^+ \setminus \ell^1, \leq^*)$ where c_0^+ is the set of all sequences of positive reals that tend to zero and ℓ^1

the set of all sequences of reals whose sum converges. The order relation \leq^* is almost domination, i.e. $\bar{f} \leq^* \bar{g}$ if $\{n : g_n > f_n\}$ is finite. The investigation of this ordering was initiated by P. Vojtáš in [11]. $(c_0^+ \setminus \ell^1, \leq^*)$ is not separative but we will show the separative quotient is isomorphic to the set \mathcal{I}_Σ of all summable ideals ordered by inverse inclusion.

We check it has the BT-property. Let us verify atomlessness. Let I be a summable ideal determined by a sequence $(a_n)_{n=0}^\infty$, and let $A \in I$. Then $\sum_{i \in \omega \setminus A} a_i$ diverges; we divide $\omega \setminus A$ into two infinite subsets B_1 and B_2 such that the appropriate sums both diverge. We make new sequences $(b_n)_{n=0}^\infty$ and $(c_n)_{n=0}^\infty$ so that $b_i = a_i$ for $i \in A \cup B_1$ and $b_i = z_i$ for $i \in B_2$, where $(z_n)_{n=0}^\infty$ is an arbitrary converging sequence of positive reals. $(c_n)_{n=0}^\infty$ is defined similarly, just B_1 and B_2 change their roles. Both $(b_n)_{n=0}^\infty$ and $(c_n)_{n=0}^\infty$ diverge. We denote the appropriate summable ideals I_b and I_c . It is clear that $I_b, I_c \supseteq I$ and that they are disjoint.

Let $(I_j)_{j \in \omega}$ be an increasing (in inclusion) sequence of summable ideals. Let $(a_n^j)_{n=0}^\infty$ be the sequence of positive reals that determines the ideal I_j . We may assume that $(a_n^0)_{n=0}^\infty \geq (a_n^1)_{n=0}^\infty \geq \dots$. Let n_0 be such that $\sum_{j \leq n_0} a_j^0 > 1$. We set $a_n = a_n^0$ for $n \leq n_0$. Then we find a $n_1 > n_0$ such that $\sum_{j=n_0+1}^{n_1} a_j^1 > 1$ and set $a_n = a_n^1$ for $n_0 < n \leq n_1$. And so on to obtain the whole sequence $(a_n)_{n \in \omega}$ so that $(a_n)_{n \in \omega} \leq^* (a_n^j)_{n \in \omega}$ for all $j \in \omega$.

To verify separativeness, consider ideals I_a and I_b , corresponding sequences $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$, such that $I_a \not\subseteq I_b$, i.e. there is a set $B \in I_b$ which does not belong to I_a . That means $\sum_{k \in B} b_k < \infty$ but $\sum_{k \in B} a_k = \infty$. If $\omega \setminus B$ belongs to I_a then I_a and I_b are already disjoint, if this is not that case then we make a new sequence $(c_n)_{n=0}^\infty$ such that $c_n = a_n$ for $n \in B$ and $\sum_{k \in \omega \setminus B} c_k < \infty$. The corresponding ideal I_c is below I_a and disjoint with I_b .

It is easy to check that if $(a_n)_{n=0}^\infty \approx_{\text{sep}} (b_n)_{n=0}^\infty$, i.e. $\forall (c_n)_{n=0}^\infty \in (c_0^+ \setminus \ell^1, \leq^*) ((c_n)_{n=0}^\infty \perp (a_n)_{n=0}^\infty \Leftrightarrow (c_n)_{n=0}^\infty \perp (b_n)_{n=0}^\infty)$, then $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$ determine the same summable ideal and the mapping $\Phi : (c_0^+ \setminus \ell^1, \leq^*) \rightarrow (\mathcal{I}_\Sigma, \supseteq)$, defined as $\Phi((c_n)_{n=0}^\infty) = \{A \subseteq \omega : \sum_{n \in A} c_n < \infty\}$, is an onto homomorphism of orderings preserving the disjointness relation. And the preimage of each summable ideal is precisely an equivalence class of sequences in \approx_{sep} .

Proposition 4.1. $\mathfrak{t}((c_0^+ \setminus \ell^1, \leq^*)) = \mathfrak{t}$.

Proof. Let $(\bar{a}_\alpha)_{\alpha < \kappa}$ be a descending chain of sequences from $(c_0^+ \setminus \ell^1, \leq^*)$ of length $\kappa < \mathfrak{t}$. We use the methods from [5] to show it has a lower bound.

For each $\alpha < \kappa$ let $h_\alpha : \omega \rightarrow \omega$ be a function such that $\forall n \in \omega (\frac{1}{h_\alpha(n)} \leq \bar{a}_{\alpha,n})$. Since $\kappa < \mathfrak{t} \leq \mathfrak{b}$, there is a function $h \in \omega^\omega$ that almost dominates all h_α 's, i.e. $h \geq^* h_\alpha$ for all $\alpha < \kappa$.

Similarly, for each $\alpha < \kappa$ let $f_\alpha : \omega \rightarrow \omega$ be a function such that $\forall n \in \omega (\sum_{f_\alpha(n) \leq i < f_\alpha(n+1)} \bar{a}_{\alpha,i} > 1)$. Since $\kappa < \mathfrak{t} \leq \mathfrak{b}$, there is a function $f \in \omega^\omega$ that almost dominates all f_α 's, i.e. $f \geq^* f_\alpha$ for all $\alpha < \kappa$. Define $g \in \omega^\omega$ recursively so that $g(0) = f(0)$ and $g(n+1) = f(g(n)+1)$. Note that for every $\alpha < \kappa$ and all but finitely many n 's $\sum_{g(n) \leq i < g(n+1)} \bar{a}_{\alpha,i} > 1$ since $g(n) < f_\alpha(g(n)) < f_\alpha(g(n)+1) \leq g(n+1)$. We denote I_n the interval $[g(n), g(n+1))$.

For every n , we denote \mathcal{F}_n the set $\{F : \text{dom}(F) \rightarrow \mathbb{Q}^+ : \text{dom}(F) \subseteq I_n \wedge \text{rng}(F) \subseteq \{\frac{1}{2|I_n|}, \frac{2}{2|I_n|}, \dots, 1\} \wedge \sum_{i \in \text{dom}(F)} F(i) > \frac{1}{2}\}$. Let $\mathfrak{F} = \bigcup_{n \in \omega} \mathcal{F}_n$. In the following we shall treat \mathfrak{F} as ω .

For every \bar{a}_α , let X_α be the set $\{F : \exists n (F \in \mathcal{F}_n \wedge \forall i \in \text{dom}(F) (F(i) \leq \bar{a}_i))\}$. An easy pigeon-hole type argument shows it is infinite for every $\alpha < \kappa$. It is also clear that $X_\beta \setminus X_\alpha$ is finite for $\alpha < \beta$. Since $\kappa < \mathfrak{t}$, there is a lower bound $X \subseteq \mathfrak{F}$. By reducing if necessary, we can assume that $|X \cap \mathcal{F}_n| \leq 1$ for every n . Finally, we define a sequence \bar{a} as follows:

For every $m \in \omega$, if there exists $F \in X$ such that $m \in \text{dom}(F)$ then we set $\bar{a}_m = F(m)$. Otherwise, we set $\bar{a}_m = \frac{1}{h(m)}$. It is now easy to check that \bar{a} is the desired lower bound.

To prove the converse, let us at first prove that $\mathfrak{t}((c_0^+ \setminus \ell^1, \leq^*)) \leq \mathfrak{b}$. Suppose the contrary. Let $(b_\alpha)_{\alpha < \mathfrak{b}}$ be a system of almost increasing functions from ω^ω without an upper bound, $\pi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ a bijection and $(l_n)_{n=0}^\infty$ a strictly decreasing sequence from ℓ^1 such that $l_n < \frac{1}{n}$ for every n . We define a descending chain of sequences from $(c_0^+ \setminus \ell^1, \leq^*)$ $(\bar{a}_\alpha)_{\alpha < \mathfrak{b}}$ as follows: $\bar{a}_{0,\pi(1,k)} = l_k$ for $k \leq b_0(0)$, for $l > b_0(0)$ we set $\bar{a}_{0,\pi(1,l)} = \frac{1}{l}$; generally, $\bar{a}_{0,\pi(n,k)} = l_k$ for $k \leq b_0(n-1)$, for $l > b_0(n-1)$ we set $\bar{a}_{n,\pi(n,l)} = \frac{1}{l}$. \bar{a}_α for other α is defined in the same way.

Let \bar{a} be a lower bound for this chain. Define a function f by $f(n) = \min\{k : \bar{a}_{\pi(n,k)} > l_k\}$. It is easy to check that f almost dominates $(b_\alpha)_{\alpha < \mathfrak{b}}$, a contradiction.

Now assume that $\mathfrak{t} < \mathfrak{t}((c_0^+ \setminus \ell^1, \leq^*))$. Let $(X_\alpha)_{\alpha < \mathfrak{t}} \subseteq [\omega]^\omega$ be a descending chain without a lower bound. We define $f_\alpha \in \omega^\omega$ for every $\alpha < \mathfrak{t}$ so that $f_\alpha(n) = k$ such that $|X_\alpha \cap [f_\alpha(n-1), f_\alpha(n))| \geq n+1$. Since $\mathfrak{t} < \mathfrak{t}((c_0^+ \setminus \ell^1, \leq^*)) \leq \mathfrak{b}$, by the already used method we find $g \in \omega^\omega$ such that for every $\alpha < \mathfrak{t}$ and for almost all n 's $|X_\alpha \cap [g(n-1), g(n))| \geq n+1$.

Define a chain $(\bar{a}_\alpha)_{\alpha < \mathfrak{t}}$ of sequences as follows: $\bar{a}_{\alpha,n} = \frac{1}{k}$ if $n \in X_\alpha \cap [g(k-1), g(k))$; if no such k exists then let $\bar{a}_{\alpha,n} = l_n$.

Finally, let \bar{a} be a lower bound for this descending chain and define a lower bound $X = \{n : \bar{a}_n > l_n\}$ for the chain $(X_\alpha)_{\alpha < \mathfrak{t}}$. \square

4.3. Meager and null ideals. Next we consider the set of all meager ideals \mathfrak{M} and the set of all ideals \mathfrak{N} of measure zero; i.e. those ideals that are meager sets and null sets respectively in the Cantor space topology. Simultaneously, we study the set of all hereditary meager and null ideals, where an ideal I is hereditary meager (null) if for every $X \in I^+$ the restriction $I \restriction X = \{A \in I : A \subseteq X\}$ is meager (null) in the Cantor space 2^X .

It is obvious they are both σ -closed. We show they are atomless, what their separative quotient is and that there is no dense subset of these orderings that has cardinality \mathfrak{c} . In fact, there is $2^{\mathfrak{c}}$ mutually disjoint elements in both orderings.

We will use the following characterization of meager ideals.

Proposition 4.2 (Talagrand; see for example Theorem 4.1.2 [4]). *An ideal I is meager if and only if there a partition $(P_i)_{i \in \omega}$ of ω into finite sets such that $\bigcup_{i \in A} P_i \in I$ iff A is finite.*

Proposition 4.3. *There are mappings $\Phi : (\mathfrak{M}, \supseteq) \rightarrow (\mathfrak{M}, \supseteq)$ and $\Psi : (\mathfrak{N}, \supseteq) \rightarrow (\mathfrak{N}, \supseteq)$ such that $\forall X \in \mathfrak{M} \forall Y \in \mathfrak{N} (\Phi(X) \supseteq X \wedge \Phi(X) \approx_{\text{sep}} X \wedge \Psi(Y) \supseteq Y \wedge \Psi(Y) \approx_{\text{sep}} Y)$.*

Proof. For a meager ideal I consider the set $\tilde{I} = \{A \subseteq \omega : I \restriction A \text{ is not meager}\}$. Let $(P_n)_{n \in \omega}$ be the partition of ω witnessing it is meager. \tilde{I} is a hereditary meager ideal containing I . To see that it is meager check that $(P_n)_{n \in \omega}$ still works. Let $A \in \tilde{I}^+$ be arbitrary. Since $A \notin \tilde{I}$ we have $I \restriction A$ is meager, so there is a partition $(Q_n)_{n \in \omega}$ of A into finite sets such that $\bigcup_{i \in C} Q_i \in I$ iff C is finite. If $\tilde{I} \restriction A$ were not meager then there would be an infinite set $C \subseteq \omega$ such that $B = \bigcup_{i \in C} Q_i \in \tilde{I} \restriction A$. $I \restriction B$ would have to be nonmeager but then there would be an infinite set $D \subseteq C$ such that $\bigcup_{i \in D} Q_i \in I \restriction A$, a contradiction.

For a null ideal I consider the set $\tilde{I} = \{A \subseteq \omega : I \restriction A \text{ is not null}\}$. To show it has measure zero consider the following subset $P = \{(A, B) : A \subseteq \omega \wedge B \in I \restriction A\}$ of $2^\omega \times 2^\omega$. P has measure zero and it follows from Fubini theorem that $\tilde{I} = \{A : P_A \text{ is not null}\}$ has measure zero. It is easy to check that \tilde{I} is downward close in inclusion and use Fubini theorem again to check it is closed under finite unions proving it is a null ideal.

Set $I_0 = I$, $I_1 = \tilde{I}_0$, $I_{n+1} = \tilde{I}_n$ and $\bar{I} = \bigcup_{n \in \omega} I_n$. It is a null ideal and moreover it is hereditary null since for every $A \in \bar{I}^+$ $\bar{I} \restriction A = \bigcup_{n \in \omega} I_n \restriction A$ is a countable union of null sets. \square

Corollary 4.4. *$(\mathfrak{M}, \supseteq)$ and $(\mathfrak{N}, \supseteq)$ are atomless, not separative, their separative quotient is isomorphic to the ordering $(\mathfrak{M}_{\mathfrak{H}}, \supseteq)$ of all hereditary meager ideals via the mapping Φ and $(\mathfrak{N}_{\mathfrak{H}}, \supseteq)$ the ordering of all hereditary null ideals via Ψ , respectively.*

Proof. To prove they are atomless, let X be an arbitrary meager ideal, let A and B be two infinite subsets of ω such that $A \cup B = \omega$ and neither A nor B is in $\Phi(X)$ ($\Phi(X)$ is meager, thus not maximal). Extend X by A and by B to obtain two disjoint ideals X_A and X_B that are easily verified to be meager. The proof for null ideals is the same.

We claim they are not separative. Consider some maximal ideal M on odd natural numbers and the ideal F of finite sets on even numbers. Then let $I = \{A \cup B : A \in M \wedge B \in F\}$ and $J = \{A \cup B : A \text{ is an arbitrary subset of odd natural numbers} \wedge B \in F\}$ be two ideals, both easily verified to be meager and null. But $\Phi(I) = \Phi(J)$ and $\Psi(I) = \Psi(J)$ which proves the claim.

On the other hand, if X and Y are two hereditary meager (null) ideals such that $X \not\supseteq Y$, then there is an infinite set $A \in Y$ that is not in X . Let Z be an ideal generated by $X \cup \{\omega \setminus A\}$, it is clearly meager and $\Phi(Z) \supseteq X$ is disjoint with Y .

It remains to show that Φ and Ψ define isomorphisms between separative quotients of $(\mathfrak{M}, \supseteq)$ and $(\mathfrak{N}, \supseteq)$ and orderings $(\mathfrak{M}_{\mathfrak{H}}, \supseteq)$ and $(\mathfrak{N}_{\mathfrak{H}}, \supseteq)$.

Φ and Ψ obviously preserve the inclusion relation and the disjointness between ideals, it suffices to prove that $\Phi(X) \approx_{\text{sep}} X$ and $\Psi(Y) \approx_{\text{sep}} Y$ for any $X \in \mathfrak{M}$ and $Y \in \mathfrak{N}$. But obviously if a meager ideal I is compatible with a meager ideal J , then $\Phi(I)$ is compatible with $\Phi(J)$ and so also with J . The same holds for null ideals. Each equivalence class of meager (null) ideals has its minimal element, the corresponding hereditary meager (null) ideal. \square

Finally, we show there is no dense subset of these orderings with size \mathfrak{c} , thus preventing them to have the BT-property.

Theorem 4.5. *There are $2^{\mathfrak{c}}$ ideals that are both meager and null and they are mutually disjoint (in both $(\mathfrak{M}, \supseteq)$ and $(\mathfrak{N}, \supseteq)$).*

In particular, neither $(\mathfrak{M}, \supseteq)$ nor $(\mathfrak{N}, \supseteq)$ has the BT-property.

Proof. Let $(I_n)_{n \in \omega}$ be a partition of ω into intervals such that $|I_n| = n + 1$. For any $X \subseteq \omega$ let X^I be the set $\bigcup_{m \in X} \{k \in \omega : \exists n (k \text{ is the } m\text{-th element of } I_n)\}$.

It is clear that if J is an ideal on ω then $\{X^I : X \in J\}$ is a base of an ideal; we shall denote this ideal as \mathcal{I}_J .

Now let \mathcal{M} be the set of all maximal ideals on ω , its size is $2^{\mathfrak{c}}$. For any $J \in \mathcal{M}$, we make an ideal \mathcal{I}_J and obtain a system \mathfrak{J} of $2^{\mathfrak{c}}$ ideals. The disjointness of two ideals $J_1, J_2 \in \mathcal{M}$ is easily seen to be preserved for \mathcal{I}_{J_1} and \mathcal{I}_{J_2} .

Claim 1 \mathfrak{J} is a system of meager ideals.

The interval partition $(I_n)_{n \in \omega}$ works for all ideals from \mathfrak{J} . Assume that some set $X \in \mathcal{I}_J$, where $\mathcal{I}_J \in \mathfrak{J}$, contains a union of infinitely many intervals. It is easy to check that once it contains the whole interval I_n then it contains all previous intervals. Thus, we conclude that $X = \omega$ which is a contradiction.

Claim 2 \mathfrak{J} is a system of null ideals.

We use characterization of null ideals (resp. filters) from [4] Theorem 4.1.3. For $m \leq n$, let i_n^m be the m -th element of I_n . Let $A_n = \{a_n = \{i_m^n : n \leq m \leq 2n - 1\}\}$. These sets satisfy the first three conditions from the theorem. Let $X \in \mathcal{I}_J$, where $\mathcal{I}_J \in \mathfrak{J}$, be a given set. $X \subseteq Y^I$ for some $Y \in J$. Then it is easy to check that $A_n \cap X = \emptyset$ for $n \in \omega \setminus Y$ and $|\omega \setminus Y| = \omega$; thus we also verified the last fourth condition and proved that every \mathcal{I}_J is null. \square

5. ITERATIONS AND PRODUCTS

If P and Q are two orderings with the BT-property then their product $P \times Q$ has again the BT-property and the height is less or equal to the minimum of heights of the original orderings. To see this, just realize that if B is a regular subalgebra of C then $\mathfrak{h}(B) \geq \mathfrak{h}(C)$, and that $\text{RO}(P)$ is a regular subalgebra of $\text{RO}(P \times Q)$. The same holds for countable products and iterations. For iterations (or products) of length greater than ω we restrict on countable support and we have to ensure that $\prod_{\alpha < \kappa}^{\text{cs}} P_\alpha$ has size \mathfrak{c} , where P_α 's are the orderings or their dense subsets witnessing the BT-property.

However, if $\kappa \geq \omega_1$ then the height of such an iteration is ω_1 .

Fact 5.1. *Let P_κ be an iteration of length $\kappa \geq \omega_1$ where P_1 has the BT-property and for every isolated $\alpha > 1 \Vdash_{P_\beta} \dot{Q}_\alpha$ has the BT-property, where $\alpha = \beta + 1$. Then assuming the condition on the size is satisfied the iteration has the BT-property and has the height ω_1 .*

Proof. Atomlessness is clear, σ -closedness folklor. Let $A_1 \subseteq P_1$ be some maximal antichain. For $\alpha = \beta + 1$ and $\alpha < \omega_1$, let $\Vdash_{P_\beta} \dot{A}_\alpha$ is a maximal antichain in \dot{Q}_α .

We let $D_1 = \{p \in P_\kappa : \exists a \in A_1(p(0) \leq a)\}$ and for other $\alpha = \beta + 1$ $D_\alpha = \{p \in P_\kappa : p \restriction \beta \Vdash \exists \dot{a} \in \dot{A}_\alpha(p(\beta) \leq \dot{a})\}$. D_α 's are dense open sets and any element from the intersection would have to have a support of size at least ω_1 . \square

Proposition 5.2. *For the ordering $([\omega]^\omega, \subseteq^*)^\alpha$, where $\alpha \leq \omega$, there is a dense subset D such that $\forall d \in D \forall i, j \in \text{dom}(d)(i \neq j \Rightarrow d(i) \cap d(j) = \emptyset)$.*

Proof. It is easy when $\alpha < \omega$. For $\alpha = \omega$, let $f \in ([\omega]^\omega, \subseteq^*)^\omega$ be given. We look for some infinite subset $d(0) \subseteq f(0)$ such that

$$(1) \quad \forall n > 0 (|f(n) \setminus d(0)| = \omega)$$

Let $\{a_\gamma : \gamma < \omega_1\}$ be some AD system of size ω_1 on $f(0)$. For every $n > 0$ there is at most one γ such that $|f(n) \setminus a_\gamma| < \omega$. Thus there is some a_δ that satisfies the condition (1). We set $d(0) = a_\delta$ and subtract $d(0)$ from $f(n)$ for every $n > 0$. We continue similarly and finally obtain $d = \langle d(n) : n \in \omega \rangle \in D$ below f . \square

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